Introduction to Constraint Programming

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many thanks to Emmanuel Hebrard

SAT/SMT/AR/CP Summer School 2022

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- The minimum about solving methods to allow for clever modeling
  - It turns out, it is already a lot!

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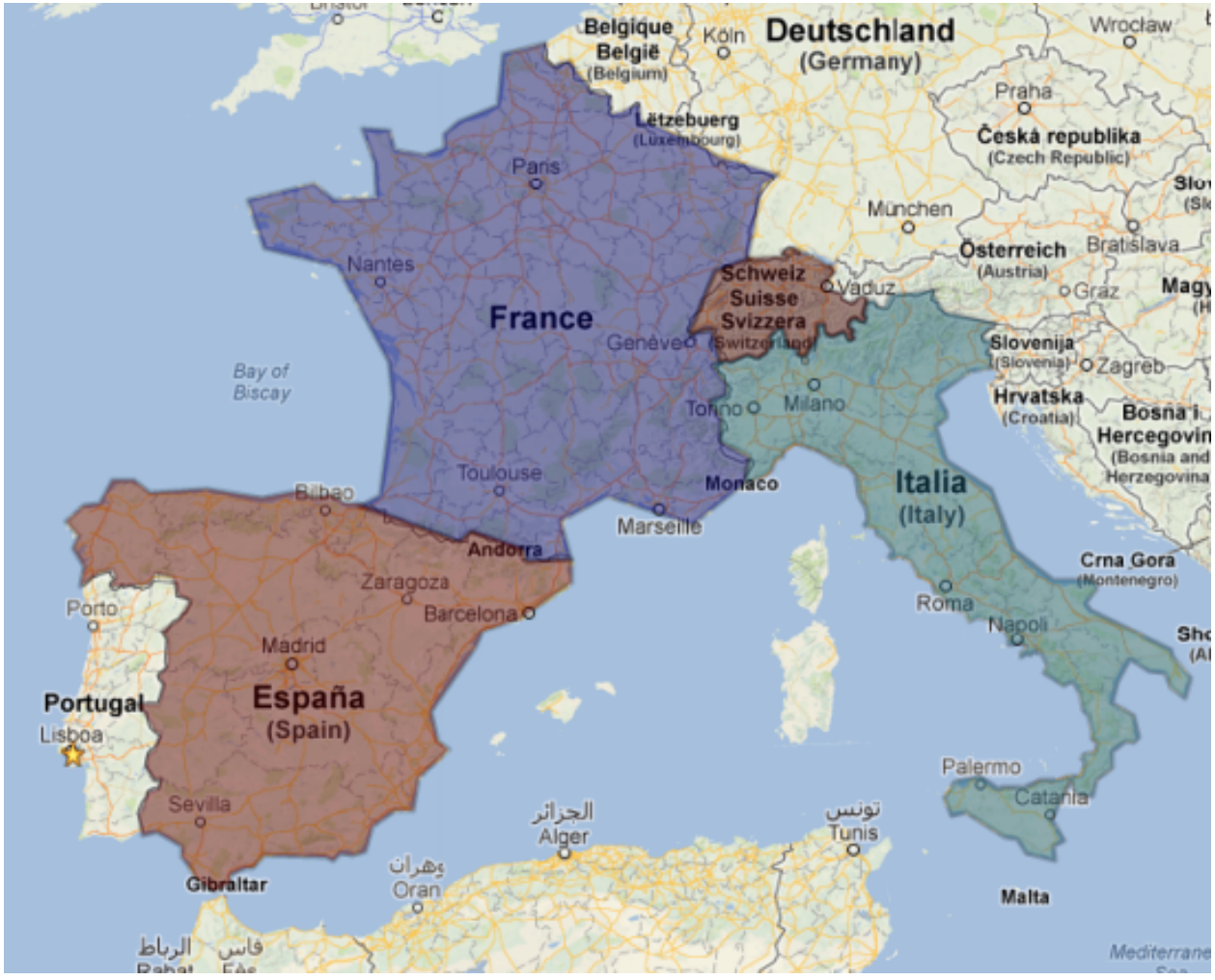
- Objective: distinguished variable to minimize/maximize

Map Coloring



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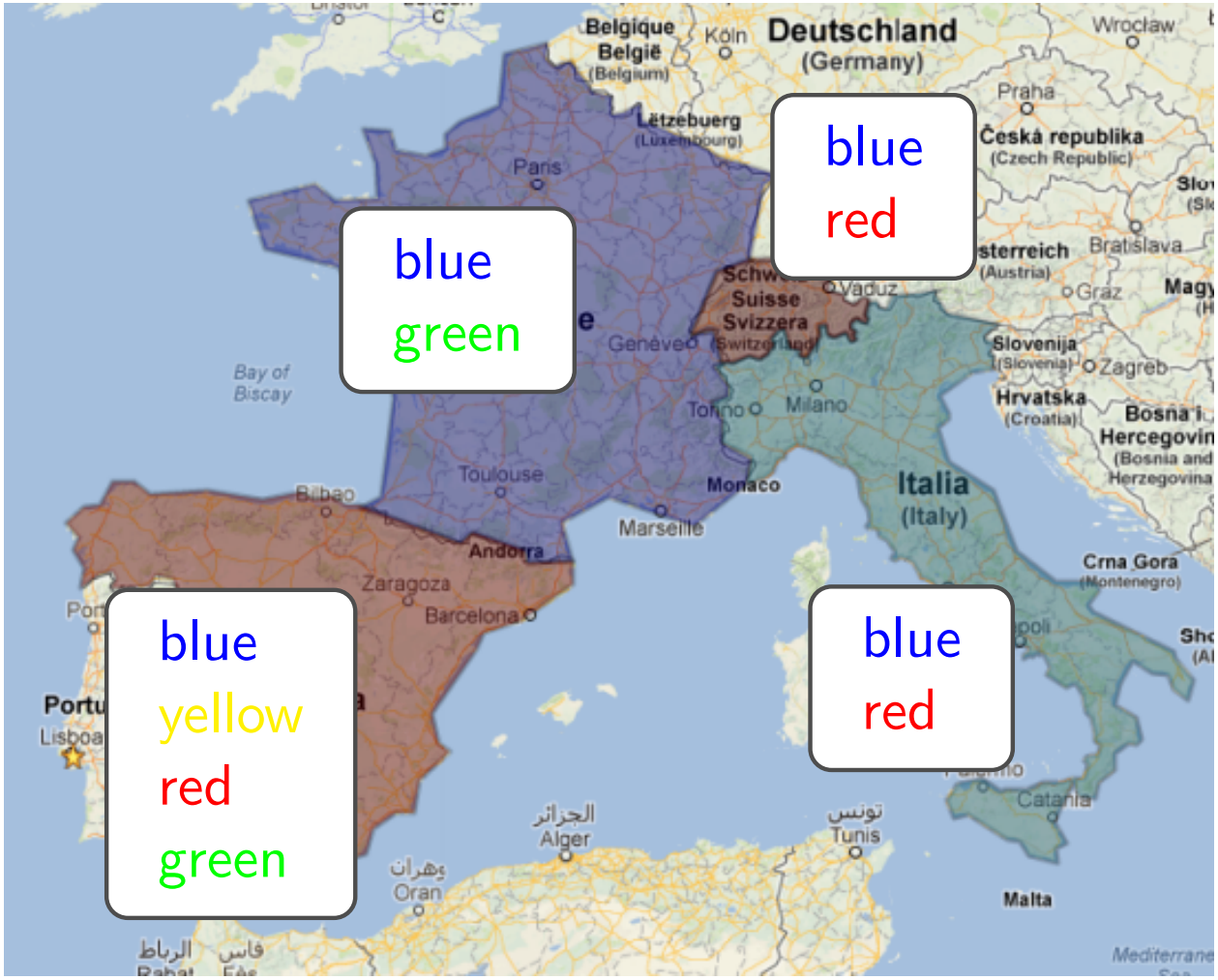

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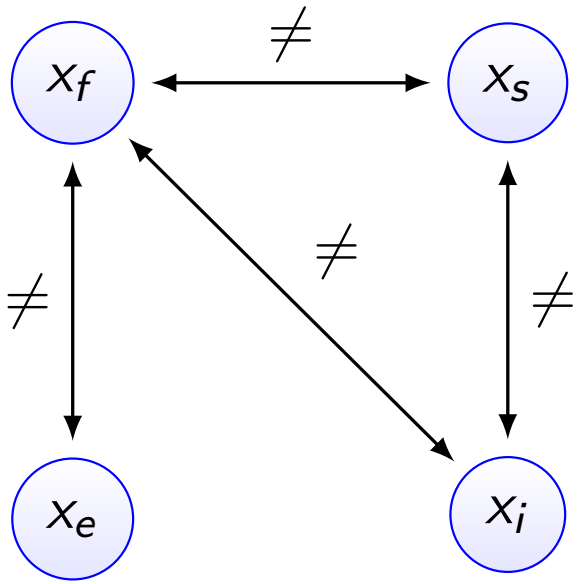
$\mathcal{D}(x_f)$: blue, green

$\mathcal{D}(x_s)$: blue, red



$\mathcal{D}(x_e)$: blue, yellow, red, green

$\mathcal{D}(x_i)$: blue, red

Map Coloring

```
france = Variable(['blue','green'], 'france')
switzerland = Variable(['blue','red'], 'switzerland')
spain = Variable(['blue','yellow','red','green'], 'spain')
italy = Variable(['blue','red'], 'italy')

model = Model(
    france != switzerland,
    france != italy,
    france != spain,
    italy != switzerland
    )
```

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  - Any Expression tree of the above

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- Consider the graph coloring example

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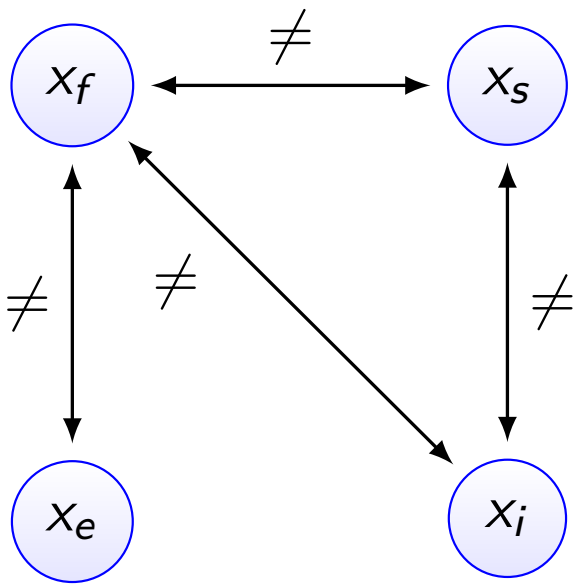

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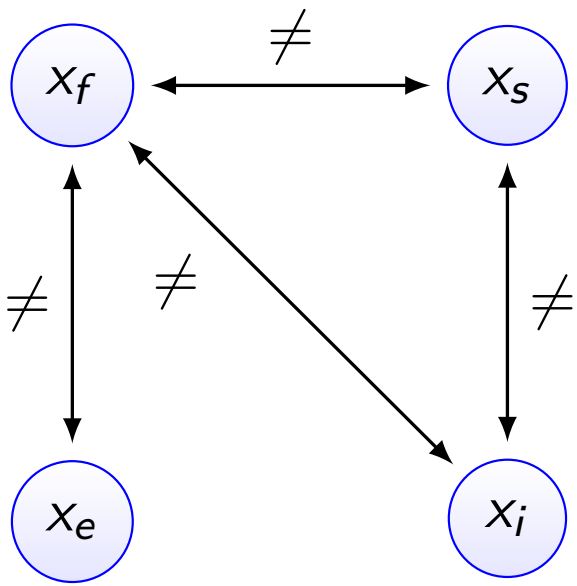

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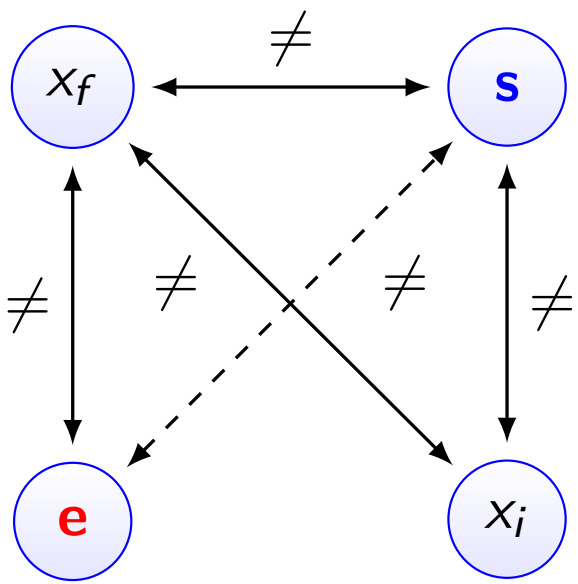

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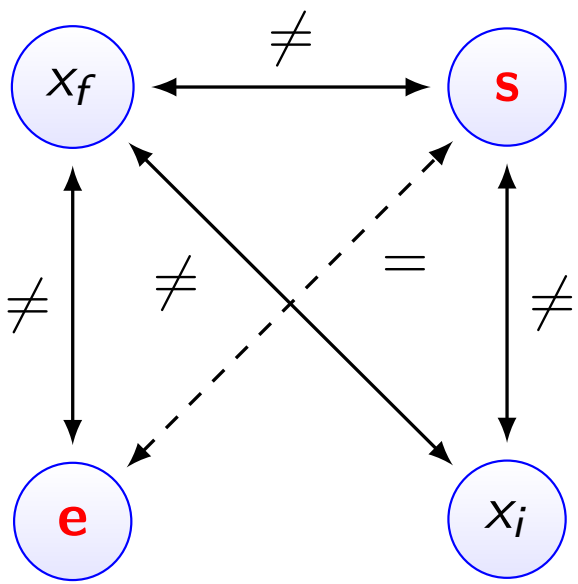

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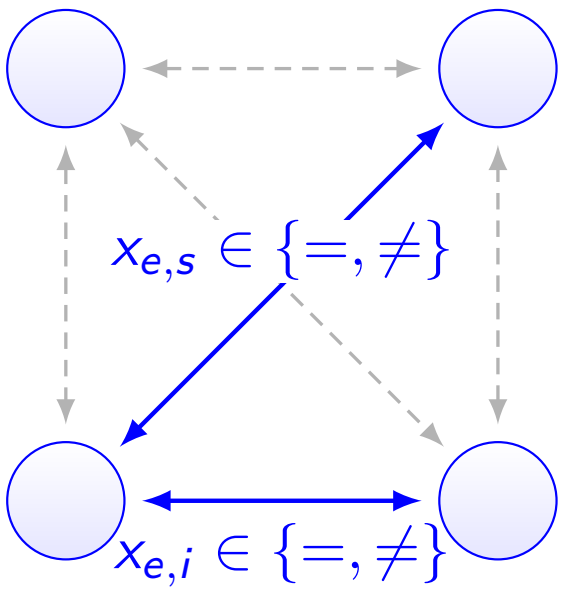

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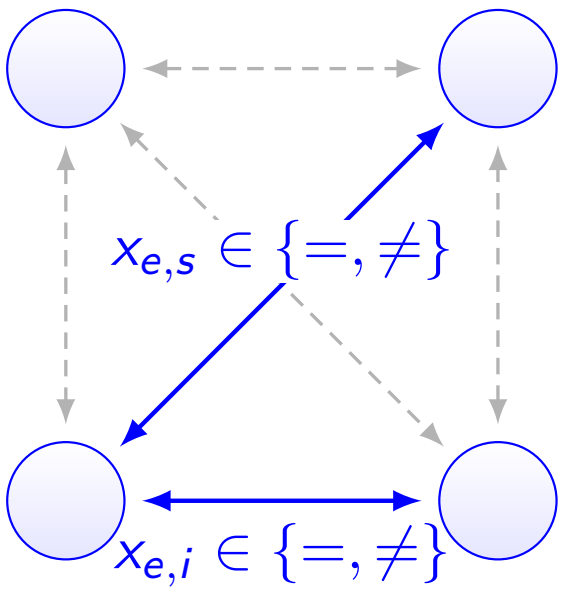

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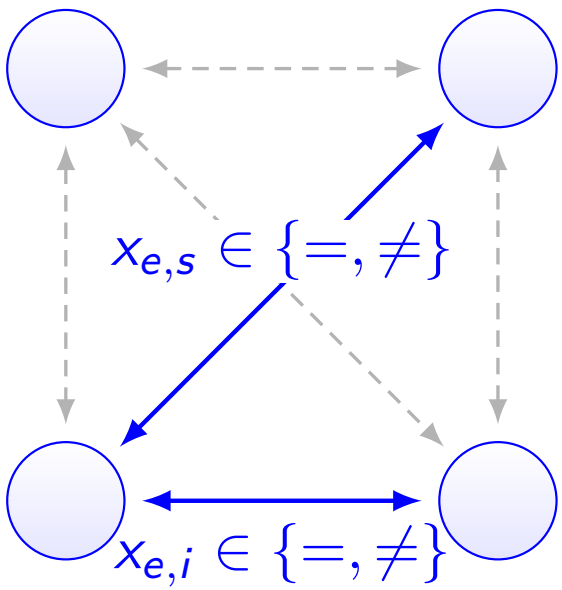

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  - either e and s take a different color, so adding the edge would not hurt
  - or e and s take the same color, so merging them (adding an equality constraint) would not hurt
- Instead of assigning colors to nodes, we can assign $\{=, \neq\}$ to non-edges
- No color symmetry anymore!
- But stating the constraints is difficult

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What is a good constraint set?

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- Two different constraints: $x \neq y$ and $(x \neq y) \iff z$ (reification)

$$(x \neq y) \implies y \leq 12 \quad \text{encoded as} \quad \begin{array}{l} (x \neq y) \iff z \\ z \implies (y \leq 12) \end{array}$$

- Which you can write $(x \neq y) \implies y \leq 12$ (and let the system insert extra variables)

Combining constraints (functionally)

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- There are also function operators that must be combined similarly
  - For instance $(|x - y| * z) \leq (z + 12)$

$$(|x - y| * z) \leq (z + 12) \quad \text{encoded as} \quad \begin{array}{l} (x - y) = a_1 \\ |a_1| = a_2 \\ a_2 * z = a_3 \\ z + 12 = a_4 \\ a_3 \leq a_4 \end{array}$$

Expression Tree

Constraints - Root of the expression tree

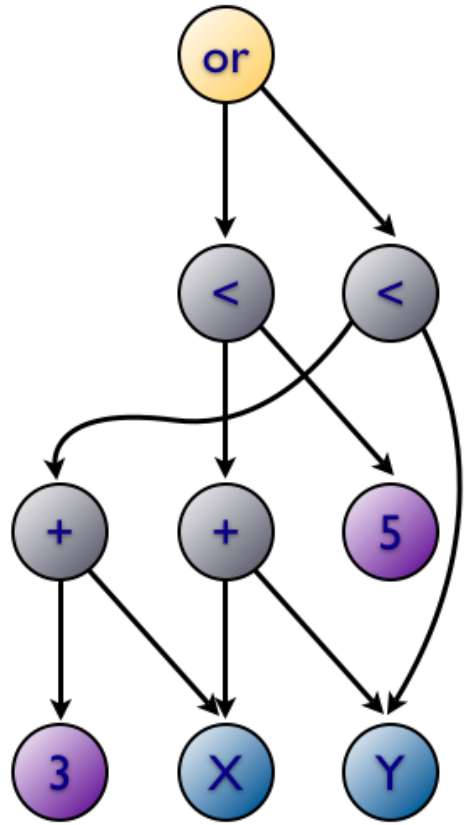
```
C1 = (X+Y < 5) | (X+3 < Y)
C2 = AllDiff([x,y,z])
C3 = Sum([a,b,c,d]) >= e
```

Predicates & functions - Internal nodes

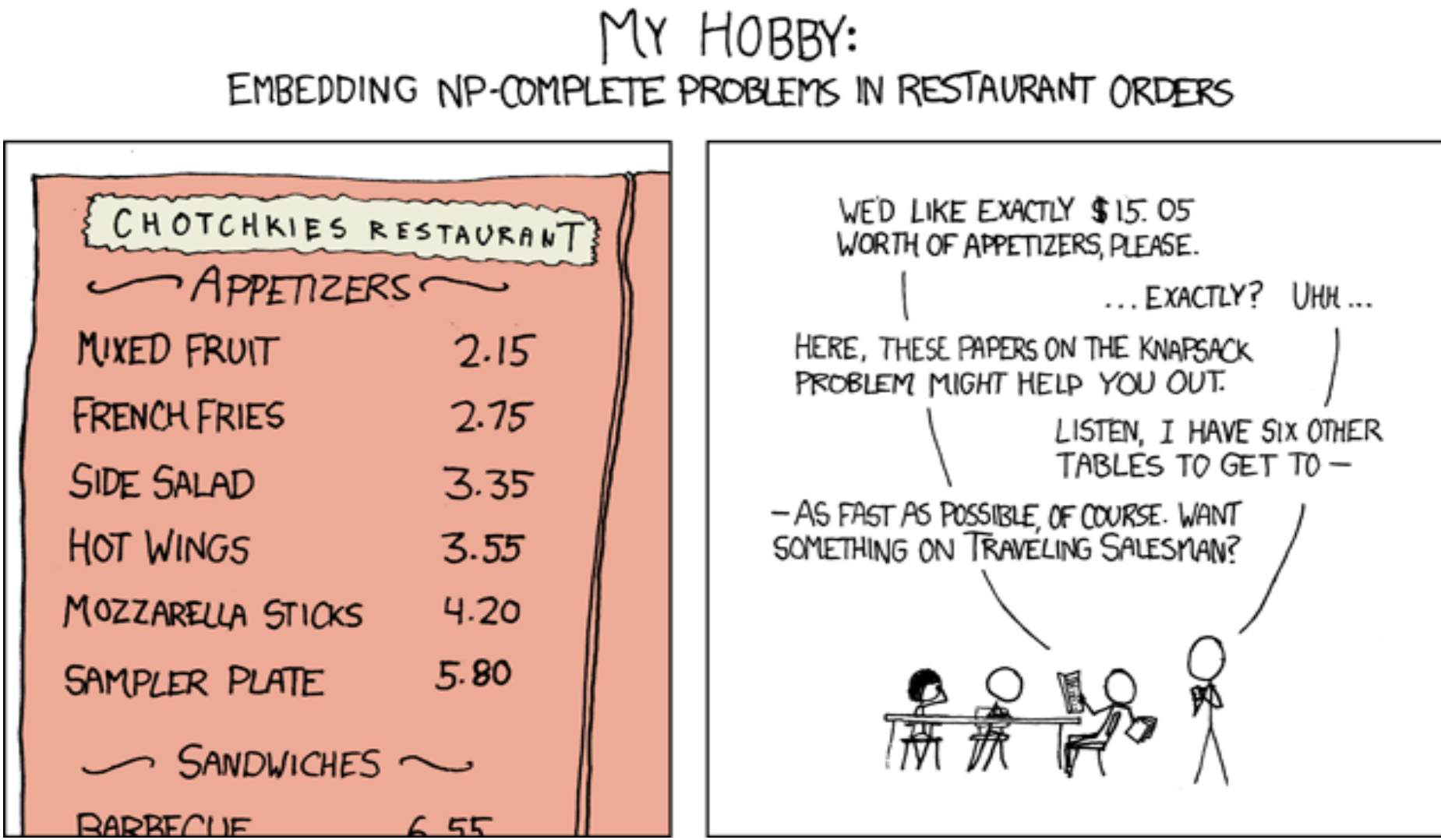
```
P = X+Y          # arythmetic value
Q = X+3 <= Y     # truth (logic) value
```

Variables - Leaves of the expression tree

```
X = Variable(0,10)
X = Variable([1,3,5,7])
```



XKCD Knapsack



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```
Sum([quantities[i] * price[i] for i in range(num_appetizers)]) == total
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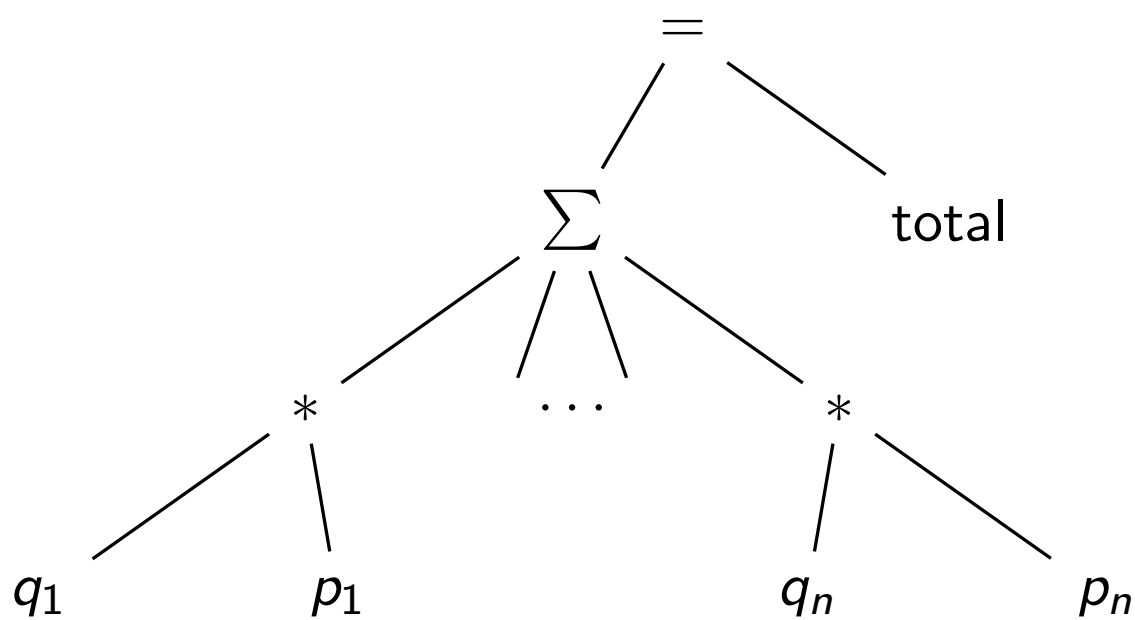

Solution

- Solution 1:

7	×	Mixed Fruit	($2.15)
0	×	French Fries	($2.75)
0	×	Side Salad	($3.35)
0	×	Hot Wings	($3.55)
0	×	Mozzarella Sticks	($4.20)
0	×	Sample Plate	($5.80)

- Solution 2:

1	×	Mixed Fruit	($2.15)
0	×	French Fries	($2.75)
0	×	Side Salad	($3.35)
2	×	Hot Wings	($3.55)
0	×	Mozzarella Sticks	($4.20)
1	×	Sample Plate	($5.80)

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$\bar{x} = 3, 5, 1, 2, 7$ satisfies AllDifferent

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Element

$$Element(x_0, \ldots, x_{n-1}, y, z) \iff x_y = z$$

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$$\mathit{Element}(x_0, \ldots, x_{n-1}, y, z) \iff x_y = z$$

$\bar{x} = 3, 5, 1, 2, 5, y = 1, z = 5$ satisfies Element

$\bar{x} = 3, 5, 1, 2, 5, y = 2, z = 5$ does not satisfy Element

Worst named constraint ever

Element

It's just array access!

$z = x[y]$

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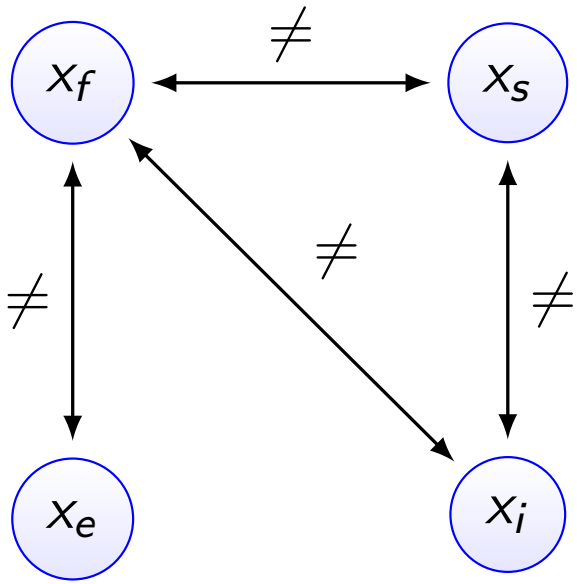

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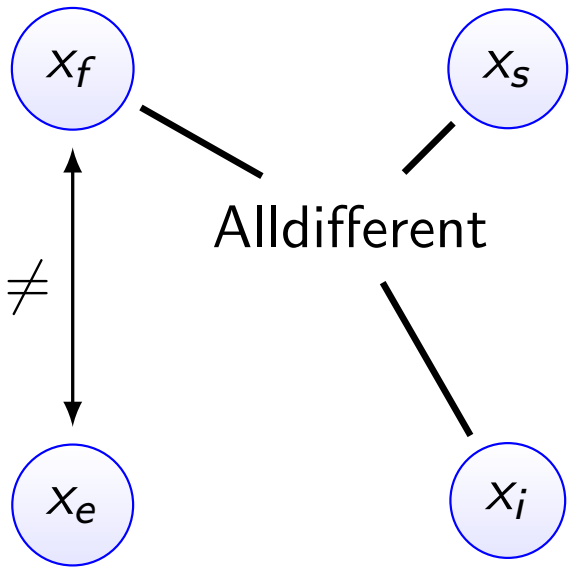

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Develop a search tree (depth first).

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constraint propagation

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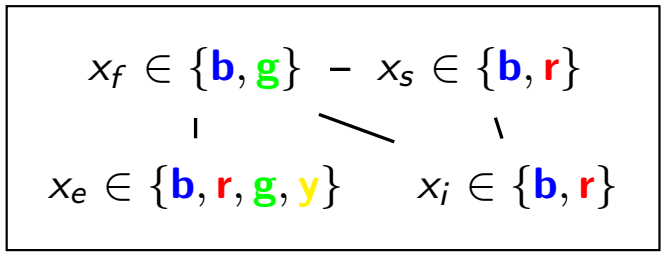
- If $\mathcal{D}(x_f) = \{v\}$ then x_e cannot take the value v
- $x_f \in \{\mathbf{b}\}, x_e \in \{\mathbf{b}, \mathbf{r}, \mathbf{g}\} \implies x_f \in \{\mathbf{b}\}, x_e \in \{\mathbf{r}, \mathbf{g}\}$

Search Tree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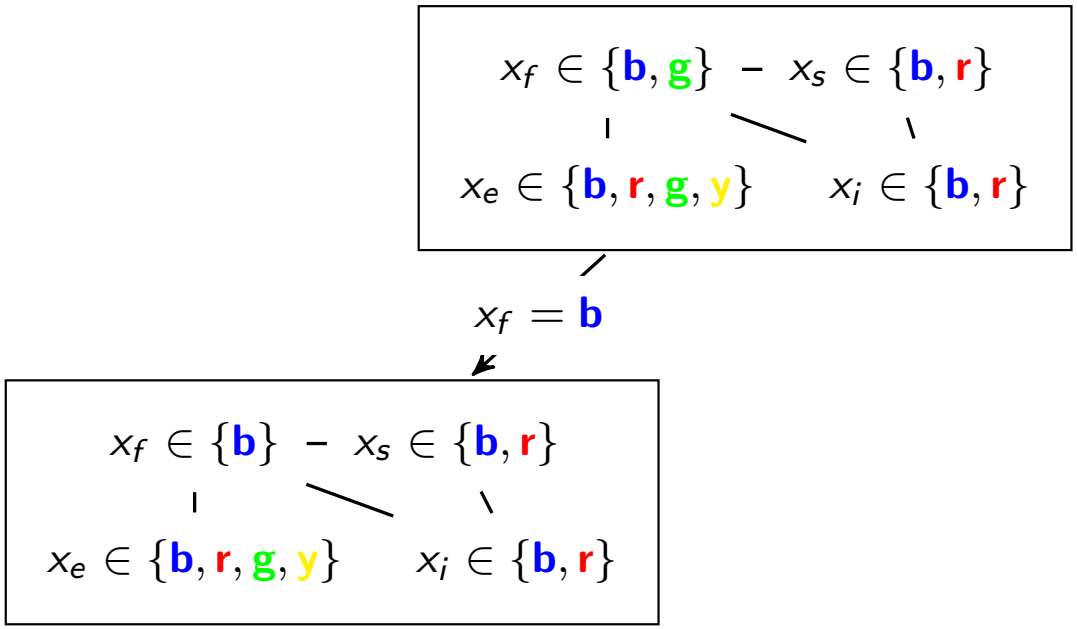
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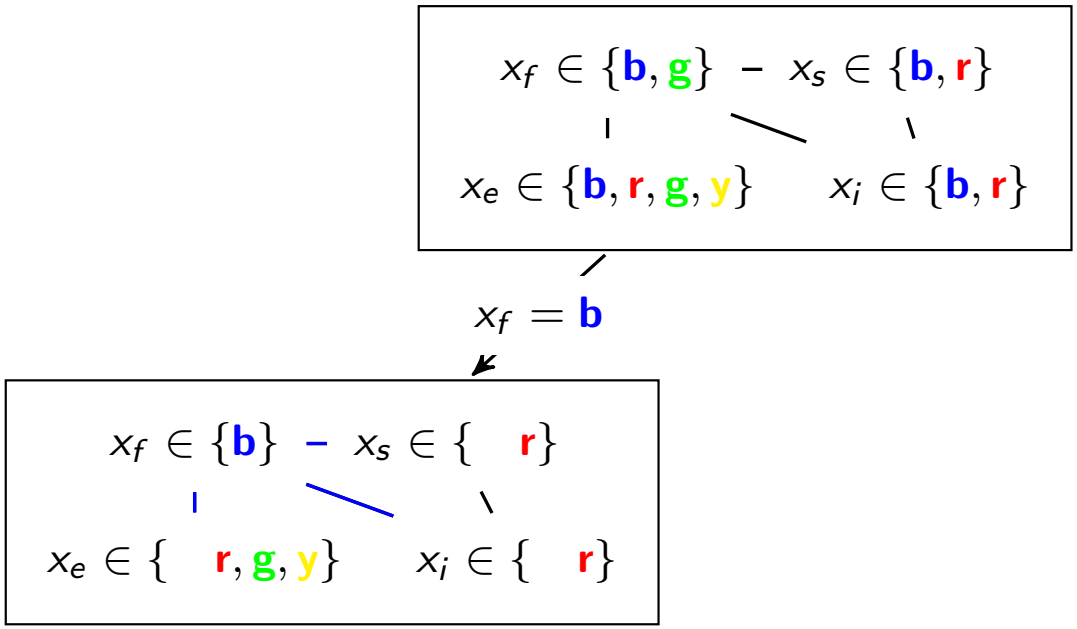
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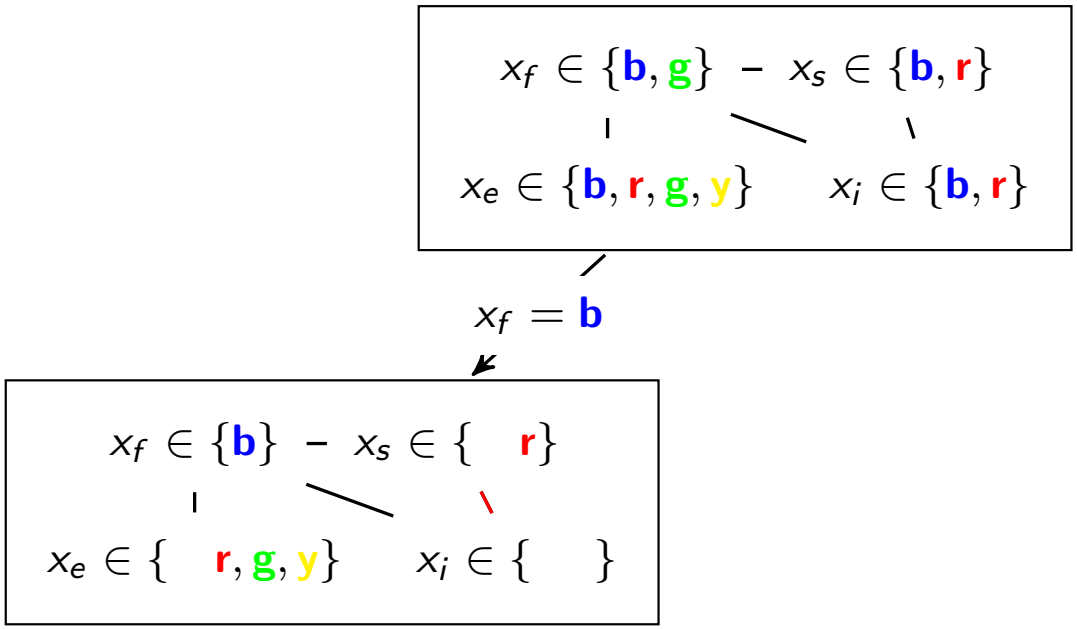
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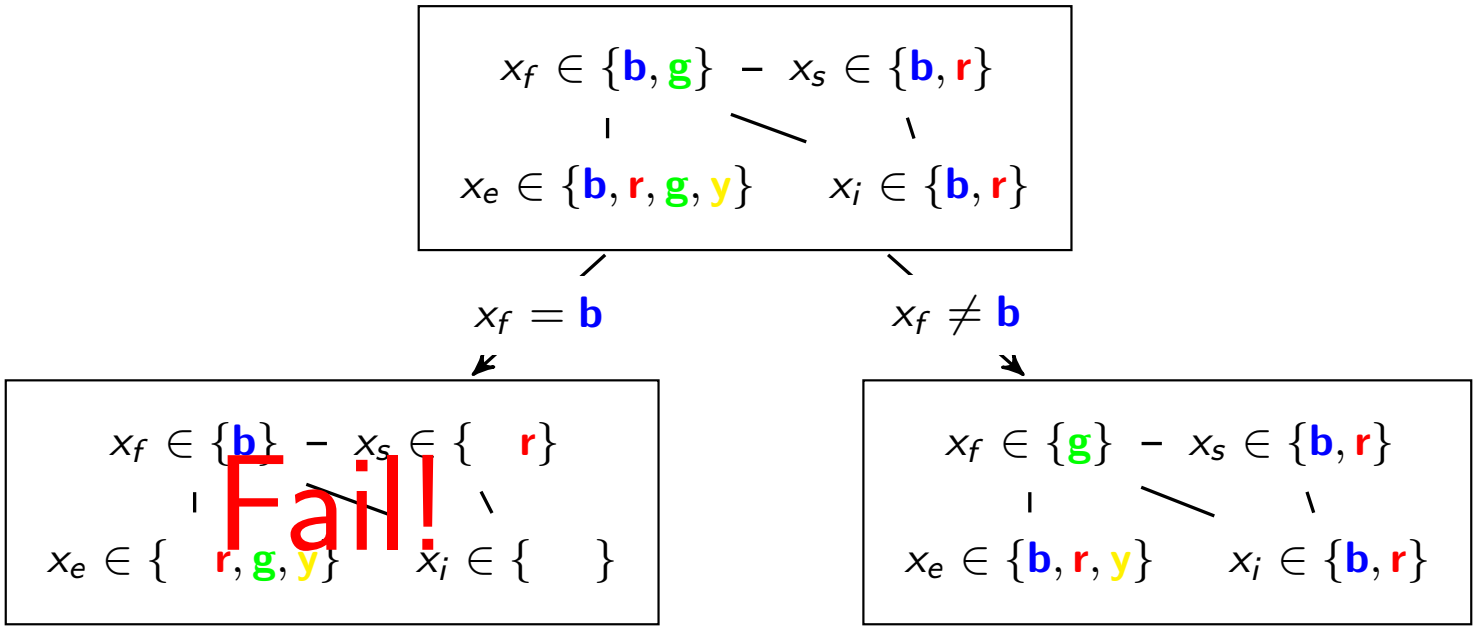
$x_e \in \{\ \mathbf{r}, \mathbf{g}, \mathbf{y}\}$
$x_i \in \{\ \}$

Search Tree


$x_f \in \{\mathbf{b}, \mathbf{g}\}$ – $x_s \in \{\mathbf{b}, \mathbf{r}\}$
$x_e \in \{\mathbf{b}, \mathbf{r}, \mathbf{g}, \mathbf{y}\}$
$x_i \in \{\mathbf{b}, \mathbf{r}\}$
$x_f = \mathbf{b}$
$x_f \neq \mathbf{b}$
$x_f \in \{\mathbf{b}\}$ – $x_s \in \{\quad \mathbf{r}\}$
$x_e \in \{\quad \mathbf{r}, \mathbf{g}, \mathbf{y}\}$
$x_i \in \{\quad\}$
Fail!
$x_f \in \{\mathbf{g}\}$ – $x_s \in \{\mathbf{b}, \mathbf{r}\}$
$x_e \in \{\mathbf{b}, \mathbf{r}, \mathbf{y}\}$
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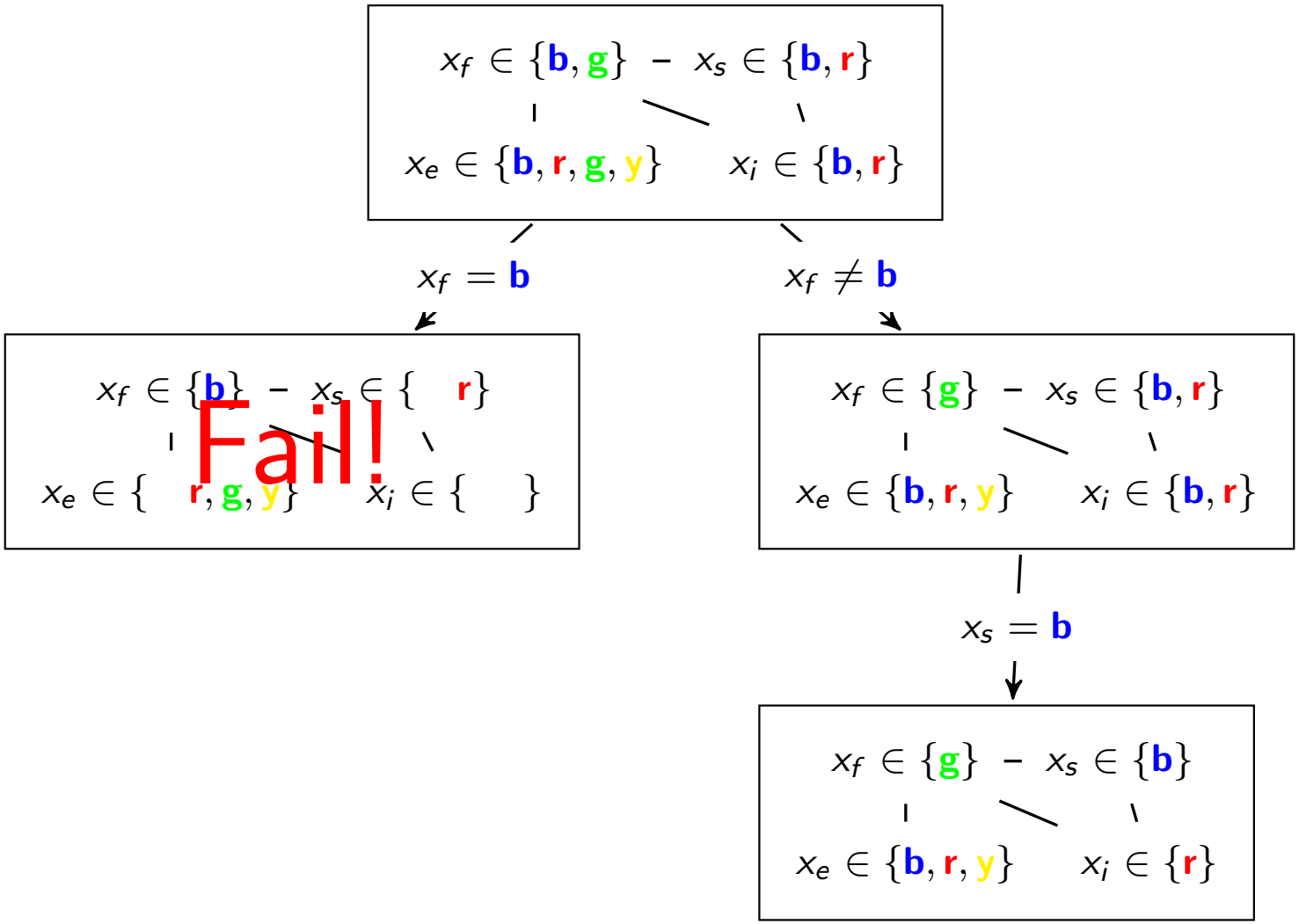
Search Tree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Example: global constraint

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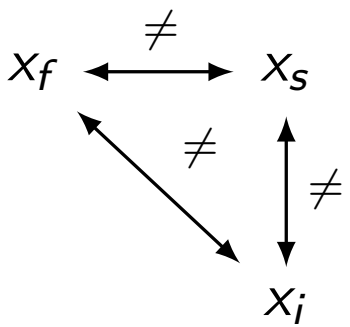

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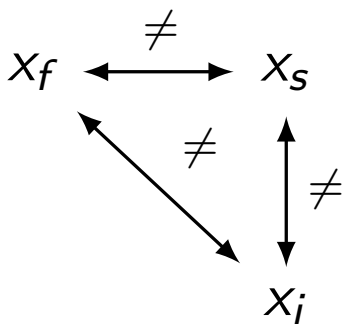

$$\begin{aligned} x_f &\in \{\mathbf{b}, \mathbf{g}\} \\ x_s &\in \{\mathbf{b}, \mathbf{r}\} \\ x_i &\in \{\mathbf{b}, \mathbf{r}\} \end{aligned}$$

- Every inequality is consistent

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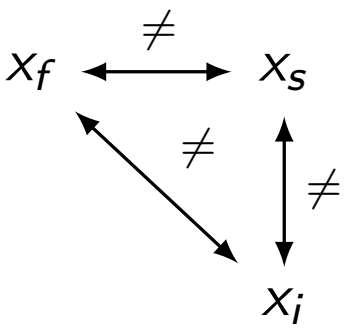

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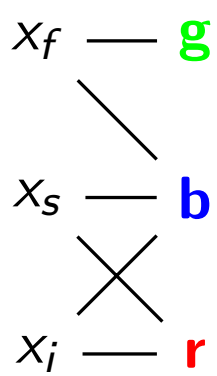
- Every inequality is consistent
- AllDifferent is not consistent!

Propagation of AllDifferent($\bar{x}$)

- A support is a perfect matching in the graph



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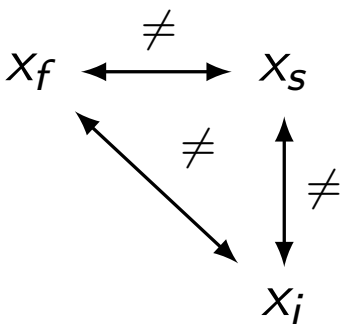

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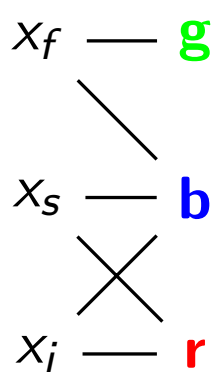
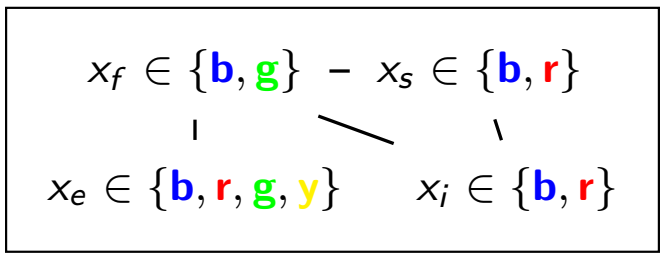
Propagation of AllDifferent($\bar{x}$)

- A support is a perfect matching in the graph
- The edge $(x_f, \mathbf{b})$ does not belong to any perfect matching
- AllDifferent(x_f, x_s, x_i) is consistent for $x_f \in \{\mathbf{g}\}$ $x_s \in \{\mathbf{b}, \mathbf{r}\}$ $x_i \in \{\mathbf{b}, \mathbf{r}\}$



Search Tree (AllDifferent)



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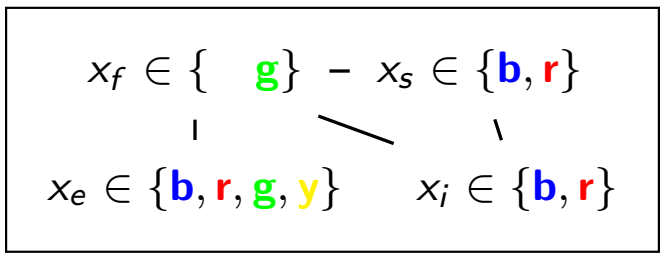

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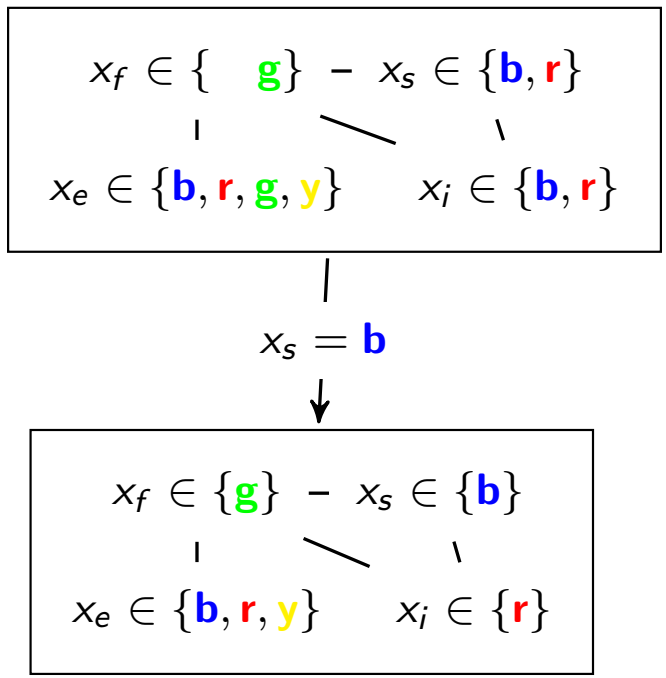

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- Every constraint has a propagation algorithm

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$\rightarrow$ Not always practical to enforce arc consistency

The art of modeling

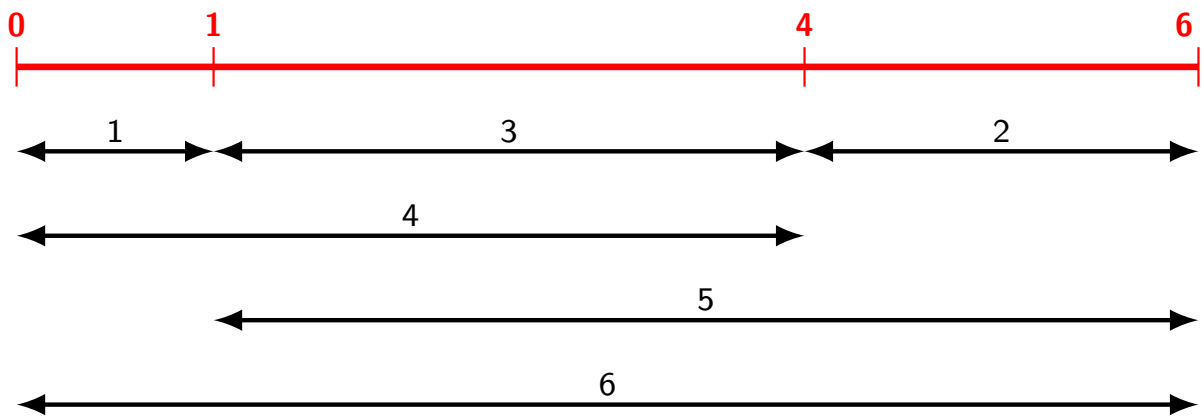
Techniques to strenghthen propagation

- Common sub-expressions
- Global constraints
- Implied constraints
- Symmetry breaking
- Dominance

Golomb Ruler

Problem definition

- Place m marks on a ruler
- Distance between each pair of marks is different
- Goal is to minimise the size of the ruler
- Proposed by Sidon [1932] then independently by Golomb and Babcock



A First Model

```
marks = VarArray(m, n, 'm')
distance = [Abs(marks[i] - marks[j]) for i in range(1, m) for j in range(i)]

model = Model(
    Minimise(Max(marks)),  # objective function

    [m1 != m2 for m1,m2 in pair_of(marks)],
    [d1 != d2 for d1,d2 in pair_of(distance)]
)
```

Branch & Bound

- An objective variable

```
model.setObjective(Model.MINIMIZE, objective);
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- The upper bound is updated when a new solution is found
- The lower bound is maintained via constraint propagation

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- Different models may entail different lower bounds for the same objective function

Global Constraints

```
marks = VarArray(m, n, 'm')
distance = [Abs(marks[i] - marks[j]) for i in range(m-1) for j in range(i+1,m)]

model = Model(
    Minimise(Max(marks)),  # objective function

    AllDiff(marks),
    AllDiff(distance)
)
```

Symmetry breaking

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- Solution symmetries $\Rightarrow$ symmetric (suboptimal) branches in the search tree

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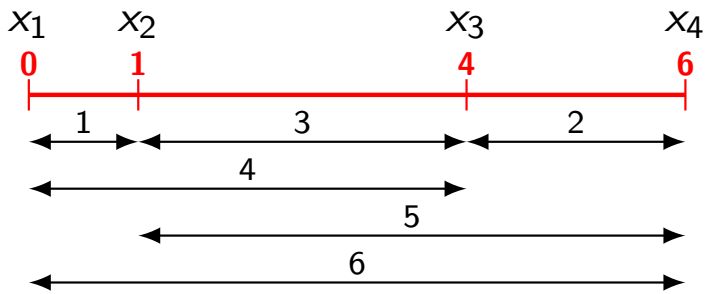

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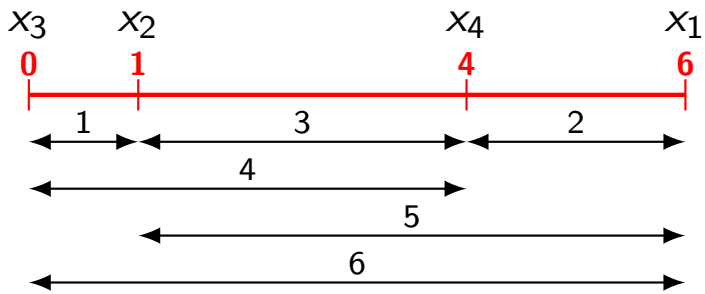

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- We can swap the marks or the distances of a solution (but not both)

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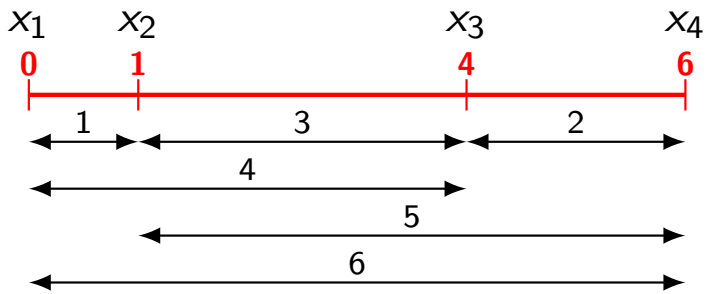

- Variable symmetries: `marks`, `distance`
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- Force an arbitrary ordering
  - `marks[1]` $<$ `marks[2]` $< \ldots <$ `marks[`m`]`

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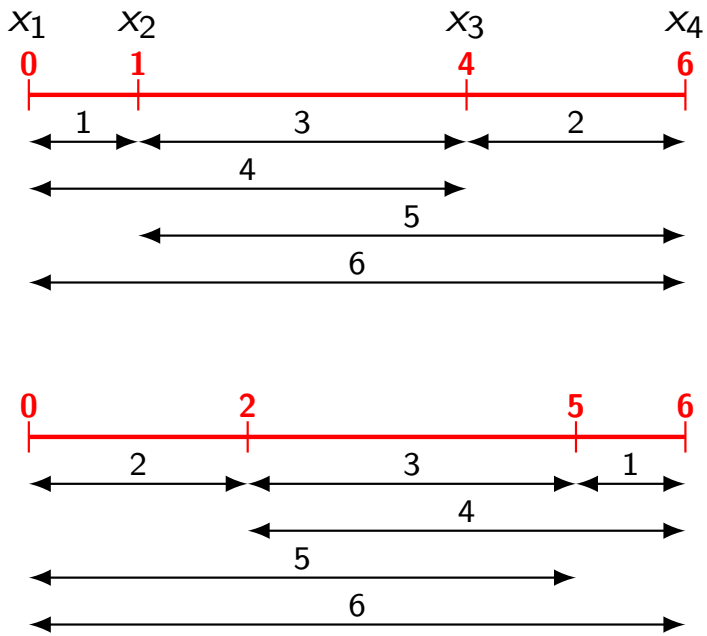

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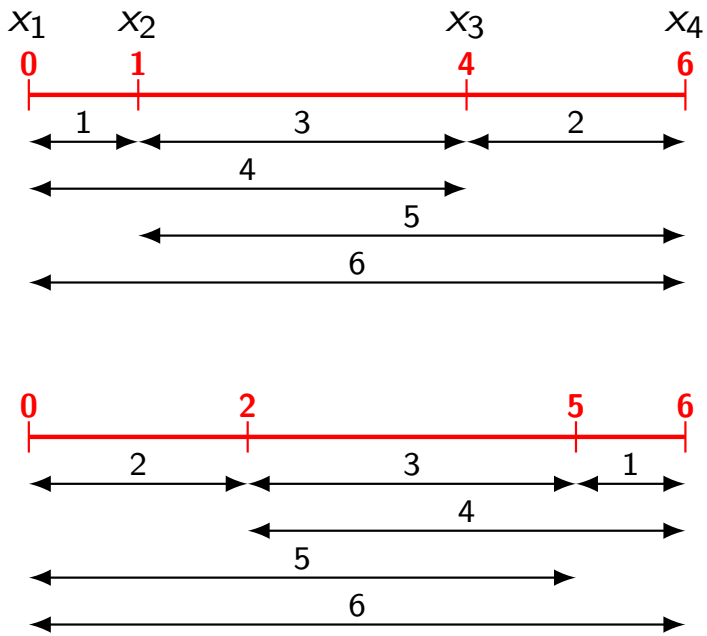

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- Force an arbitrary ordering
  - `marks[1]` $<$ `marks[2]` $< \ldots <$ `marks[`m`]`
- Distances are still symmetric by reflection
  - `distance[0,1]` $<$ `distance[`$m-2, m-1$`]`

Symmetry breaking

```
marks = VarArray(m, n, 'm')
distance = [marks[j] - marks[i] for i in range(m-1) for j in range(i+1,m)]

model = Model(
    Minimise(marks[-1]),  # objective function

    [marks[i-1] < marks[i] for i in range(1, m)],
    marks[0] == 0,
    distance[0] < distance[-1],
    AllDiff(distance)
)
```

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  - consistent with $x \in \{1, \ldots, 9\}, y \in \{2, \ldots, 10\}$

Implied Constraints: Golomb Ruler

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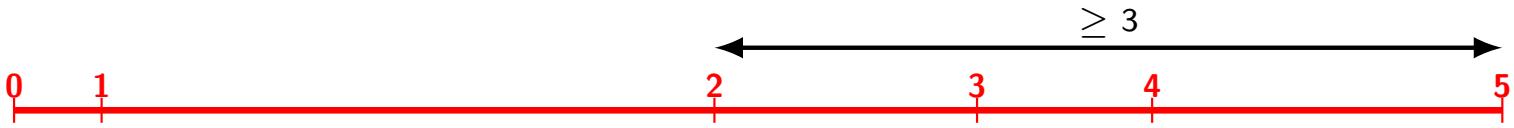
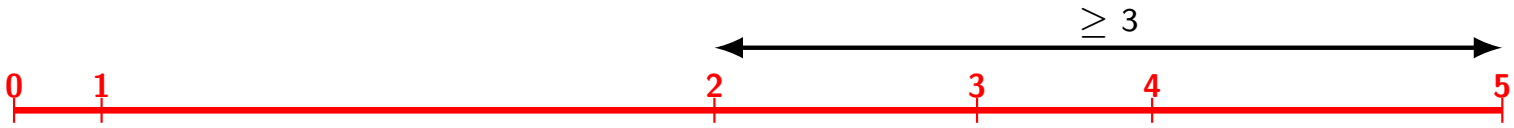


- `distance[i,j]` $\geq$ sum of $j - i$ distances

Implied Constraints: Golomb Ruler



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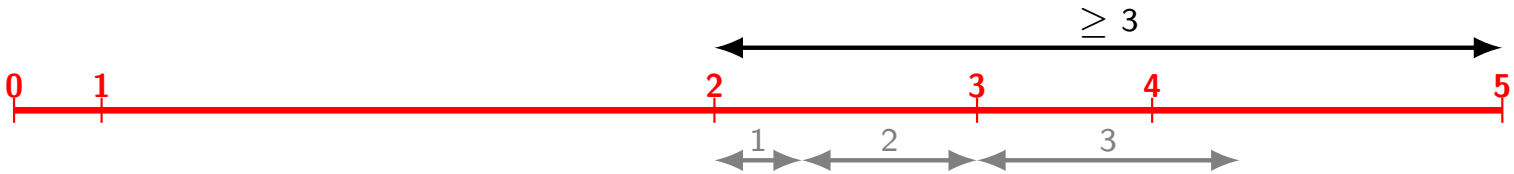

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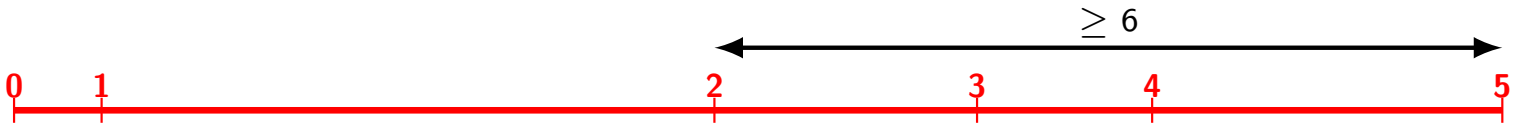

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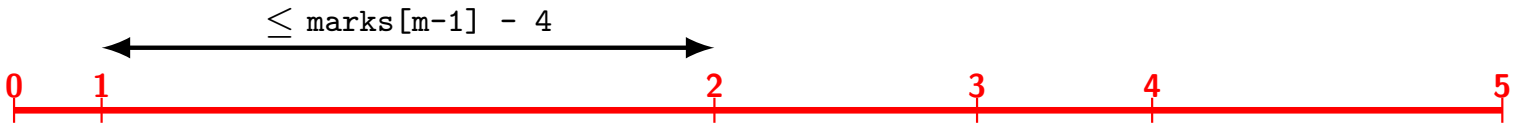

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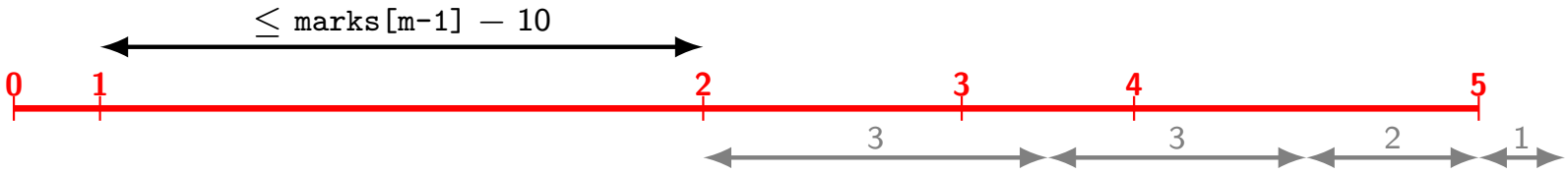

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- Only practice gives a definitive answer

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Good modeling practices

- What are the variables, what are the values?
  - Constraints will follow
  - Defines the shape of the search tree
- Key principle: **strengthen constraint propagation**
  - Global constraints
  - Implied constraints
  - Symmetry breaking